ELSEVIER Journal of Geometry and Physics 18 (1996) 195–214

JOURNAL OF GEOMETRY AND PHYSICS

Lagrangian formalism over graded algebras

Alexander Verbovetsky [1]

S.I.S.S.A., Via Beirut 2-4, 34013 Trieste, Italy

Received 3 January 1995; revised 2 March 1995

Abstract

This paper provides a description of an algebraic setting for the Lagrangian formalism over graded algebras and is intended as the necessary first step towards the noncommutative $\mathcal{C}$-spectral sequence (variational bicomplex). A noncommutative version of integration procedure, the notion of adjoint operator, Green's formula, the relation between integral and differential forms, conservation laws, Euler operator, Noether's theorem is considered.

Keywords: Graded module; Commutation factor; Adjoint operator; Berezinian complex of integral forms; Lagrangian formalism; Supermanifold; Noncommutative differential calculus

1991 MSC: 58E30, 58A50, 58B30, 81T60, 83E50, 16W50, 46L87, 14A22

0. Introduction

An outstanding progress which has in last years been made by noncommutative geometry stems out of shifting of the interests away from the original idea of geometrizing noncommutative rings using the language of noncommutative schemes. Now it has become clear that, bypassing the difficult problem of glueing noncommutative spectra, one can define directly differential geometric objects on a hypothetical "noncommutative space". This is based on two essential points: the possibility of reformulating the classical notions of analysis and differential geometry in pure algebraic terms, so that differential calculus becomes an extension of the language of commutative algebra (for a very enlightening discussion see [1]), and the existence of several quite important cases in which one is able to go beyond the commutative case (see, e.g., [2]). These ideas have proven to be very successful and give an impetus to new researchers whose aim is to transplant the classical tools of analysis and geometry into a noncommutative setting. An instance of this comes from the current work

[1] E-mail: verbovet@sissa.it.

SSDI 0393-0440(95)00017-8

on noncommutative (and first of all supercommutative) theory of integration, conservation laws, and the Lagrangian formalism (see [3–17] and others). A key problem that arises here is the description of integration procedure. The first question is the following: Given a commutative algebra, how to define the module of volume forms? An answer to this question must, in particular, give the possibility to define the Berezin volume forms on a supermanifold in usual fashion, i.e., by using the rule of signs. The well-known peculiarities of the Berezin integration show the character of the problem and result in the loss of a clear algebraic setting for integration procedure and related things.

The subject of this paper is an algebraic picture underlying the Lagrangian formalism and giving a solution of the problems discussed above. We work here over an arbitrary graded-commutative (with respect to a commutation factor) algebra to show that our constructions of volume forms, adjoint operators, the Euler operator, algebraic Green's formula, Noether's theorem, etc. can be extended to such an algebra in a simple and straightforward manner. On the other hand, the class of graded-commutative algebras is quite important for its own sake because it includes supercommutative, color-commutative (see [18–25] and references therein), and "quantum" algebras (see [26,20,27] and others). We have chosen here not to accumulate formulas for specific algebras, but to present a general scheme, using examples for illustrations only. The applications will be described separately. Our ultimate goal, which is outside the present paper, is to develop the super- and non-commutative generalizations of the $\mathcal{C}$-spectral sequence (variational bicomplex) (see [28–31]), which is a means for studying all aspects of the Lagrangian formalism: the inverse problem, the description of conservation laws, characteristic classes and so on.

The paper is organized as follows. In Section 1 we outline the necessary definitions and facts from calculus over graded algebras. In Section 2 the main objects of this paper — adjoint operators and the Berezinian [2] — are defined. We want to emphasize that these definitions, which are the *deus ex machina* from that everything else follows, arose from an interplay between ideas and constructions of work [32], where the Berezin forms have been explained in terms of $\mathcal{D}$-modules, and works [33] and [29, Part I], where structure of Lagrangian formalism for a smooth commutative algebra has been clarified. In Section 3 we consider the Spencer complexes, algebraic Green's formula, and related staff. The main difference from nongraded case (see [29]) is the appearance of a new complex dual to the de Rham complex: complex of integral forms (the name borrowed from the supergeometry). The Lagrangian formalism, theory of conservation laws, and the Noether theorem are developed in Section 4. In Appendix A we briefly describe, in a pure algebraic manner, the formalism of right connections (see [34,35]), which is closely related with our subject.

1. A sketch of differential calculus over graded algebras

1.1

We start with definitions of graded objects (see, e.g., [36]).

[2] We use this name for the module of volume forms.

Let G be an Abelian group written additively, which will serve as a grading group, and K a commutative ring with unit.

Denote by K^* the multiplicative group of invertible elements of K. Let us fix a *commutation factor* $\{\cdot,\cdot\} : G \times G \to K^*, g_1 \times g_2 \mapsto \{g_1, g_2\}$, i.e., a map satisfying two properties:

(1) $\{g_1, g_2\}^{-1} = \{g_2, g_1\}$

(2) $\{g_1 + g_2, g_3\} = \{g_1, g_3\}\{g_2, g_3\}$

We have also $\{g_1, g_2 + g_3\} = \{g_1, g_2\}\{g_1, g_3\}$ as a consequence of the definition.

Example 1.1. Let $G = \mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. Then it is easily shown that any commutation factor has the form:

$$\{g, h\} = \prod_{i=1}^{n} q_i^{g_i h_i} \cdot \prod_{i<j} q_{ij}^{(g_i h_j - h_i g_j)},$$

where $g = (g_1, \ldots, g_n), h = (h_1, \ldots, h_n) \in \mathbb{Z}^n, q_i = \pm 1, q_{ij} \in K^*$.

Example 1.2. In particular, if $G = \mathbb{Z}$ there exist two commutation factors: the trivial one $\{g_1, g_2\} = 1$ and the standard superfactor $\{g_1, g_2\} = (-1)^{g_1 g_2}$.

Example 1.3. If $G = \mathbb{Z}^n$ and $K = \mathbb{C}$ then $\{g, h\} = \prod_{i=1}^{n} (\pm 1)^{g_i h_i} \cdot e^{\theta(g,h)}$, where $\theta : G \times G \to \mathbb{C}$ is an antisymmetric bilinear form.

Suppose $A = \oplus_{g \in G} A_g$ is a G-graded associative K-algebra with unit. A is called *graded commutative* if

$$ab = \{a, b\} ba \quad \forall a, b \in A.$$

For this type of notation we always assume that a and b are homogeneous and that the symbol of graded object used as argument of the commutative factor denotes the grading of this object.

Example 1.4. A commutative algebra (graded or not) is graded commutative with respect to the trivial commutation factor $\{g_1, g_2\} = 1$.

Example 1.5. The algebra $C^\infty(M)$ of smooth functions on a supermanifold M is graded commutative with respect to the superfactor $\{g_1, g_2\} = (-1)^{g_1 g_2}, G = \mathbb{Z}_2$.

Example 1.6 *Quantum superplane* [37,2]. Let K be a field and K^n the n-dimensional coordinate vector space over K. Picking up an arbitrary commutation factor over $\mathbb{Z}^n$ (see Examples 1.1–1.3), the quadratic algebra $A = \mathrm{T}(K^n)/R$, where $\mathrm{T}(K^n)$ is the tensor algebra over K^n with the natural $\mathbb{Z}^n$-grading and R is the ideal in $\mathrm{T}(K^n)$ generated by the elements $x_1 \otimes x_2 - \{x_1, x_2\} x_2 \otimes x_1, x_1, x_2 \in K^n$, is called the algebra of polynomial functions on the quantum superplane $\mathbb{A}_q$.

Example 1.7 *Noncommutative torus* [38,2]. Let $K = \mathbb{C}, G = \mathbb{Z}^n$, and A be the space of all complex valued functions on $\mathbb{Z}^n$ that decay faster than any polynomial. A has the natural $\mathbb{Z}^n$-grading. Denote by $e(k)$ the function which is equal to 1 at $k \in \mathbb{Z}^n$ and zero at all other points (a basic harmonic on a noncommutative torus). Then any element of A can be written as $\sum_k a_k e(k)$. Given a skew bilinear form $\theta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, we define a multiplication on A by setting $e(k)e(l) = \mathrm{e}^{2\pi \mathrm{i}\theta(k,l)} e(k+l)$. It is straightforward to check that the algebra A, called the algebra of (smooth) functions on the noncommutative torus $\mathbb{T}_\theta$, is graded commutative with respect to the commutation factor from Example 1.3.

Remark 1.8. In [39,40] Lychagin has shown that one can also define commutation factors over a noncommutative group G and introduce all notions of the G-graded differential calculus in these circumstances.

In this article we work in the category $\mathcal{M}od_A$ of all G-graded modules over a graded-commutative algebra A. Clearly any left module P can be transformed canonically into a right module, $pa = \{p, a\} ap$ for $a \in A, p \in P$, so that we may consider P as a bimodule.

Remark 1.9. The category $\mathcal{M}od_A$ is a *closed tensor category* (see, e.g., [2]) with respect to the commutativity constraint

$$S_{P,Q} : P \otimes Q \to Q \otimes P, \qquad p \otimes q \mapsto \{p, q\} q \otimes p.$$

A generalization of our constructions to an arbitrary Abelian closed tensor category appears to be very interesting. Significant results along this line, dealing with basic concepts of the differential calculus, may be found in [41,42].

1.2

We now introduce basic objects of the differential calculus (for details see [43–46]).

Let $\Delta \in \mathrm{Hom}_K(P, Q)$ be a K-homomorphism, P and Q being A-modules. For every element $a \in A$ define a K-homomorphism $\delta_a(\Delta) : P \to Q$ by setting $\delta_a(\Delta)(p) = \{a, \Delta\}\Delta(ap) - a\Delta(p), p \in P$. Obviously, $\delta_a \circ \delta_b = \delta_b \circ \delta_a \ \forall a, b \in A$. Put $\delta_{a_0,\dots,a_k} = \delta_{a_0} \circ \cdots \circ \delta_{a_k}$.

Definition 1.10. A K-homomorphism $\Delta \in \mathrm{Hom}_K(P, Q)$ is called a *differential operator* (d.o.) of order $\le k$, if for all $a_0, \dots, a_k \in A$ we have $\delta_{a_0,\dots,a_k}(\Delta) = 0$.

The set of all d.o.'s of order $\le k$, from P to Q, may be endowed with two A-module structures by putting $a\Delta = a \circ \Delta$ or $a\Delta = \{a, \Delta\}\Delta \circ a$, where $a \in A$ is understood as the operator of multiplication by a. The modules that arise in this way are denoted by $\mathrm{Diff}_k(P, Q)$ and $\mathrm{Diff}_k^+(P, Q)$, respectively. Clearly, $\mathrm{Diff}_k^{(+)}(P, Q) \subset \mathrm{Diff}_l^{(+)}(P, Q)$ for $k \le l$, so that we may consider the union $\mathrm{Diff}^{(+)}(P, Q) = \bigcup_{k \ge 0} \mathrm{Diff}_k^{(+)}(P, Q)$.

Proposition 1.11.

(1) *If* $\Delta_1 \in \mathrm{Diff}_k(P, Q)$, $\Delta_2 \in \mathrm{Diff}_l(Q, R)$, P, Q, R *are* A*-modules , then* $\Delta_2 \circ \Delta_1 \in \mathrm{Diff}_{k+l}(P, R)$.

(2) *The maps* $\mathrm{Diff}_k(P, Q) \to \mathrm{Diff}_k^+(P, Q)$ *and* $\mathrm{Diff}_k^+(P, Q) \to \mathrm{Diff}_k(P, Q)$ *generated by the identity map are d.o.'s of order* $\le k$.

(3) *There exists a canonical isomorphism*

$$\mathrm{Diff}_k^+(P, Q) \to \mathrm{Hom}_A(P, \mathrm{Diff}_k^+(Q)),$$

where $\mathrm{Diff}_k^+(Q) = \mathrm{Diff}_k^+(A, Q)$. *To every d.o.* $\Delta : P \to Q$ *corresponds the homomorphism* $\varphi_\Delta \in \mathrm{Hom}_A(P, \mathrm{Diff}_k^+(Q))$, $\varphi_\Delta(p)(a) = \Delta(pa)$ *under this isomorphism. The inverse mapping takes a homomorphism* $\varphi : P \to \mathrm{Diff}_k^+(Q)$ *to an operator* $\mathcal{D}_k \circ \varphi$, *where* $\mathcal{D}_k : \mathrm{Diff}_k^+(Q) \to Q$ *is a d.o. of order* $\le k$ *defined by the formula* $\mathcal{D}_k(\nabla) = \nabla(1)$, $\nabla \in \mathrm{Diff}_k^+(Q)$.

Proof. It consists of a series of automatic verifications. □

This proposition has the following corollary.

Corollary 1.12. *The commutative diagram*

$$\begin{array}{ccc}
\mathrm{Diff}_k^+(\mathrm{Diff}_l^+(P)) & \xrightarrow{\mathcal{D}_k} & \mathrm{Diff}_l^+(P) \\
{\scriptstyle c_{k,l}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_l} \\
\mathrm{Diff}_{k+l}^+(P) & \xrightarrow{\mathcal{D}_{k+l}} & P
\end{array}$$

uniquely defines the operator $c_{k,l}$, *which is said to be glueing operator.*

Definition 1.13. A k-multilinear over K mapping $\nabla : A \times \cdots \times A \to P$, P being an A-module, is said to be a *skew multiderivation* if the following conditions hold:

(1) $\nabla(a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) = -\{a_i, a_{i+1}\}\nabla(a_1, \ldots, a_{i+1}, a_i, \ldots, a_k)$

(2) $\nabla(a_1, \ldots, a_{i-1}, ab, a_{i+1}, \ldots, a_k) = \{\nabla, a\}\{a_1 \cdots a_{i-1}, a\}a\nabla(a_1, \ldots, b, \ldots, a_k) + \{\nabla, b\}\{a_1 \cdots a_{i-1}a, b\}b\nabla(a_1, \ldots, a, \ldots, a_k)$.

The set of all skew multiderivations is an A-module denoted by $\mathrm{D}_k(P)$. Obviously $\mathrm{D}_1(P)$ is a submodule of $\mathrm{Diff}_1(A, P)$. *The Spencer* Diff*-operator* $S : \mathrm{D}_k(\mathrm{Diff}_l^+(P)) \to \mathrm{D}_{k-1}(\mathrm{Diff}_{l+1}^+(P))$ is defined by the formula

$$S(\nabla)(a_1, \ldots, a_{k-1})(a) = \nabla(a_1, \ldots, a_{k-1}, a)(1), \quad \nabla \in \mathrm{D}_k(\mathrm{Diff}^+(P)).$$

Proposition 1.14.

(1) S *is a d.o. of order* ≤ 1;

(2) $S^2 = 0$.

Proof. It is straightforward. □

The complex $0 \leftarrow P \xleftarrow{\mathcal{D}} \mathrm{Diff}^{+}(P) \xleftarrow{S} \mathrm{D}_1(\mathrm{Diff}^{+}(P)) \xleftarrow{S} \mathrm{D}_2(\mathrm{Diff}^{+}(P)) \longleftarrow \cdots$ is said to be *the Spencer* Diff*-complex.*

1.3

The operations $Q \mapsto \mathrm{D}_k(Q)$, $Q \mapsto \mathrm{Diff}(P, Q)$, and $Q \mapsto \mathrm{Diff}^{+}(Q, P)$ are functors from the category of all A-modules $\mathcal{M}od_A$ to itself. We have seen that the latter functor is representable. The following proposition shows that two other functors are also representable.

Proposition 1.15. (see [43,45,46]). *There exists a module Λ^k (resp. $\mathcal{J}^k(P)$), which is called the module of k-form (resp. k-jets) over A, such that the functor $Q \mapsto \mathrm{D}_k(Q)$ (resp. $Q \mapsto \mathrm{Diff}_k(P, Q)$) is isomorphic to the functor $Q \mapsto \mathrm{Hom}_A(\Lambda^k, Q)(resp. Q \mapsto \mathrm{Hom}_A(\mathcal{J}^k(P), Q))$.*

Let us denote by $j_k = j_k(P) : P \to \mathcal{J}^k(P)$ the d.o. of order $\leq k$ that corresponds under the isomorphism $\mathrm{Diff}_k(P, \mathcal{J}^k(P)) = \mathrm{Hom}_A(\mathcal{J}^k(P), \mathcal{J}^k(P))$ to the identical map $\mathrm{id}_{\mathcal{J}^k(P)}$. Proposition 1.15 implies that the operator j_k is universal, i.e., for every d.o. $\Delta \in \mathrm{Diff}_k(P, Q)$ there exists a unique homomorphism $\psi_\Delta : \mathcal{J}^k(P) \to Q$ such that $\Delta = \psi_\Delta \circ j_k$.

Now let $\mathcal{M}$ be a subcategory of category $\mathcal{M}od_A$ of all A-modules that is closed under the action of the above-discussed functors. Then it is quite natural to ask if these functors are representable in $\mathcal{M}$.

Example 1.16. The most important example of the closed (in the above sense) category is that of geometrical modules $\mathcal{M}od_A^g$. A module P is called geometrical if $\tilde{P} = \bigcap_{\wp,k} \wp^k P = 0$, where intersection is taken over all powers of prime ideals of A. There is the geometrization functor $P \mapsto P/\tilde{P}$ from $\mathcal{M}od_A$ to $\mathcal{M}od_A^g$. It can easily be checked that the functors D_k and $\mathrm{Diff}_k(P, \cdot)$ are representable in $\mathcal{M}od_A^g$, the geometrization functor transforming the representing objects in $\mathcal{M}od_A$ into the corresponding representing objects in $\mathcal{M}od_A^g$.

1.4

A natural transformation of the functors D_k, Diff_k, and their compositions by duality gives rise to operators between the corresponding representing objects.

Example 1.17. The natural inclusion $\mathrm{D}_{k+l}(P) \to \mathrm{D}_k(\mathrm{D}_l(P))$ defines the wedge product of forms over A: $\Lambda^k \otimes \Lambda^l \to \Lambda^{k+l}$.

Example 1.18. The Spencer operator $S : \mathrm{D}_k(P) \to \dot{\mathrm{D}}_{k-1}(\mathrm{Diff}_1^{+}(P))$, where the superscribed dot means that the K-module $\mathrm{D}_{k-1}(\mathrm{Diff}_1^{+}(P))$ is supplied with A-module structure by putting $a\theta = \mathrm{D}_{k-1}(\mathrm{Diff}_1^{+}(a))\theta, \theta \in \mathrm{D}_{k-1}(\mathrm{Diff}_1^{+}(P))$, induces the homomorphism $\mathcal{J}^1(\Lambda^{k-1}) \to \Lambda^k$. The composition of $\Lambda^{k-1} \xrightarrow{j_1} \mathcal{J}^1(\Lambda^{k-1}) \to \Lambda^k$ is called *the exterior differentiation* operator and is denoted by $d : \Lambda^{k-1} \to \Lambda^k$. Using the fact that $S^2 = 0$, one can easily prove that $d^2 = 0$. The complex $0 \to A \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \cdots$ is said to be the *de Rham complex* of the algebra A.

1.5

The above described algebraic formalism can be realized geometrically, the algebra A being the algebra $C^\infty(M)$ of smooth real functions on a supermanifold M (for the supergeometry see, e.g., [47,48,35,49,50]). In this situation it may be shown in the same way as in nongraded case that the standard notions of differential operator, forms, jets, etc. coincide with the ones introduced above. Having this in mind, in Section 2 we illustrate constructions under consideration by giving their local coordinate description.

2. Adjoint operators and Berezinian

2.1

Given an A-module P, consider the complex of homomorphisms

$$0 \to \mathrm{Diff}^+(P, A) \xrightarrow{w} \mathrm{Diff}^+(P, \Lambda^1) \xrightarrow{w} \mathrm{Diff}^+(P, \Lambda^2) \xrightarrow{w} \cdots, \tag{1}$$

where $w(\nabla) = d \circ \nabla \in \mathrm{Diff}^+(P, \Lambda^k)$, $\nabla \in \mathrm{Diff}^+(P, \Lambda^{k-1})$. Let us denote the cohomology module in a term $\mathrm{Diff}^+(P, \Lambda^n)$ by $\widehat{P}_n, n \geq 0$. Every d.o. $\Delta : P \to Q$ generates the natural map of the complexes:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Diff}^+(Q, \Lambda^{k-1}) & \xrightarrow{w} & \mathrm{Diff}^+(Q, \Lambda^k) & \longrightarrow \cdots \\
& \tilde{\Delta}\big\downarrow & & \tilde{\Delta}\big\downarrow & \\
\cdots \longrightarrow & \mathrm{Diff}^+(P, \Lambda^{k-1}) & \xrightarrow{w} & \mathrm{Diff}^+(P, \Lambda^k) & \longrightarrow \cdots,
\end{array}$$

where $\tilde{\Delta}(\nabla) = \{\Delta, \nabla\}\nabla \circ \Delta \in \mathrm{Diff}^+(P, \Lambda^k), \nabla \in \mathrm{Diff}^+(Q, \Lambda^k)$.

Definition 2.1. The operator $\Delta_n^* : \widehat{Q}_n \to \widehat{P}_n$ induced by $\tilde{\Delta}$ is called the (nth) *adjoint operator*.

Below we assume that an integer n is fixed and omit the corresponding index to simplify notation.

Proposition 2.2.
(1) *Δ^* has the same grading as Δ.*
(2) *If $\Delta \in \mathrm{Diff}_k(P, Q)$ then $\Delta^* \in \mathrm{Diff}_k(\widehat{Q}, \widehat{P})$.*
(3) *For all $\Delta_1 \in \mathrm{Diff}(P, Q)$ and $\Delta_2 \in \mathrm{Diff}(Q, R)$ we have*

$$(\Delta_2 \circ \Delta_1)^* = \{\Delta_2, \Delta_1\}\Delta_1^* \circ \Delta_2^*.$$

Proof.

(1) Obvious.

(2) Denote by $[\nabla]$ the cohomologous class of an operator $\nabla \in \mathrm{Diff}^+(P, \Lambda^n), w(\nabla) = 0$. Then $\Delta^*([\nabla]) = \{\Delta, \nabla\}[\nabla \circ \Delta]$ and we have $\delta_a(\Delta^*)([\nabla]) = \{a, \Delta^*\}\Delta^*(a[\nabla]) - a\Delta^*([\nabla]) = \{a, \Delta^*\}\{a, \nabla\}\{\Delta, \nabla \circ a\}[\nabla \circ a \circ \Delta] - \{\Delta, \nabla\}\{a, \nabla \circ \Delta\}[\nabla \circ \Delta \circ a] = (a \circ \Delta)^*([\nabla]) - \{a, \Delta\}(\Delta \circ a)^*([\nabla]) = -\delta_a(\Delta)^*([\nabla])$, i.e., $\delta_a(\Delta^*) = -\delta_a(\Delta)^*$. Thus $\delta_{a_0,\dots,a_k}(\Delta^*) = (-1)^{k+1}\delta_{a_0,\dots,a_k}(\Delta)^* = 0$.

(3) $(\Delta_2 \circ \Delta_1)^*([\nabla]) = \{\Delta_2 \circ \Delta_1, \nabla\}[\nabla \circ \Delta_2 \circ \Delta_1] = \{\Delta_2, \nabla\}\{\Delta_2, \Delta_1\}\Delta_1^*([\nabla \circ \Delta_2]) = \{\Delta_2, \Delta_1\}\Delta_1^*(\Delta_2^*([\nabla]))$. □

Let us consider some examples of adjoint operators.

Example 2.3. Let $a : P \to P$ be the operator of multiplication by $a \in A$. Then $a^*([\nabla]) = \{a, \nabla\}[\nabla \circ a] = a[\nabla]$, i.e., $a^* = a$.

Example 2.4. Let $p : A \to P$ be the operator $p(a) = pa, a \in A, p \in P$. Then $p^*([\nabla]) = \{p, \nabla\}[\nabla \circ p], p^* \in \mathrm{Hom}_A(\widehat{P}, \widehat{A})$. Thus we have a natural pairing $\langle\cdot,\cdot\rangle : P \otimes \widehat{P} \to \widehat{A}$, $\langle p, \widehat{p}\rangle = p^*(\widehat{p}), \widehat{p} \in \widehat{P}$.

Example 2.5 *Berezinian and integral forms.* Let $\cdots \to P_{k-1} \xrightarrow{\Delta_k} P_k \to \cdots$ be a complex of d.o.'s. Since $\Delta_k^* \circ \Delta_{k+1}^* = \{\Delta_k, \Delta_{k+1}\}(\Delta_{k+1} \circ \Delta_k)^* = 0$, we get a complex $\cdots \leftarrow \widehat{P}_{k-1} \xleftarrow{\Delta_k^*} \widehat{P}_k \leftarrow \cdots$, which is called dual to given one. The complex dual to the de Rham complex is said to be the *complex of integral forms* and is denoted by

$$0 \leftarrow \Sigma_0 \xleftarrow{\delta} \Sigma_1 \xleftarrow{\delta} \cdots,$$

where $\Sigma_i = \widehat{\Lambda^i}, \delta = d^*$. The module $\Sigma_0 = \widehat{A}$ is called the *Berezinian* (*or the module of volume forms*) and is denoted by B.

The d.o.'s $\mathcal{D} : \mathrm{Diff}^+(\Lambda^k) \to \Lambda^k$ induce the cohomology map $\int : B \to \mathrm{H}^n(\Lambda^*)$, so that to any element $\omega \in B$ (volume form) corresponds the nth de Rham cohomology class $\int \omega$. This is an algebraic version of the integration operation. Clearly, $\int[\nabla] = \nabla(1) \bmod d\Lambda^{n-1}, \nabla \in \mathrm{Diff}(A, \Lambda^n)$.

Remark 2.6. The construction and the whole line of this subsection, as it was mentioned in the Introduction, are principally motivated by Penkov's explanation of the Berezin forms on a supermanifold [32]. Close approaches to the construction of the Berezin forms on a supermanifold were suggested in [51] and [52].

Proposition 2.7. $\int \delta\omega = 0$, *where* $\omega \in \Sigma_1$.

Proof. Suppose $\omega = [\nabla]$, then $\delta\omega = [\nabla \circ d]$. Therefore $\int \delta\omega = [\nabla \circ d(1)] = 0$. □

Proposition 2.8 (*Integration by parts*). *For any d.o.* $\Delta : P \to Q$ *and* $p \in P, \widehat{q} \in \widehat{Q}$, *we have*

$$\int \langle \Delta(p), \widehat{q} \rangle = \{\Delta, p\} \int \langle p, \Delta^*(\widehat{q}) \rangle.$$

Proof. Suppose $\widehat{q} = [\nabla], \nabla : Q \to \Lambda^n$. Then $\int \langle \Delta(p), \widehat{q} \rangle = \int \{\Delta(p), \nabla\}[\nabla \circ \Delta(p)] = \int \{\Delta(p), \nabla\}[\nabla \circ \Delta \circ p] = \int \{\Delta, \nabla\}\{\Delta, p\} \langle p, [\nabla \circ \Delta] \rangle = \{\Delta, p\} \int \langle p, \Delta^*(\widehat{q}) \rangle$. □

2.2. Coordinates

Let M be a smooth supermanifold of dimension $s \mid t, G = \mathbb{Z}_2, K = \mathbb{R}$ or $\mathbb{C}, A = C^\infty_K(M), P = \Gamma(\alpha)$ and $Q = \Gamma(\beta)$ the modules of smooth sections of vector bundles over M. Suppose $x = (y_i, \xi_j), i = 1, \dots, s, j = 1, \dots, t, x_1 = y_1, \dots, x_s = y_s, x_{s+1} = \xi_1, \dots, x_{s+t} = \xi_t$ is a coordinate system on a domain $\mathcal{U} \subset M$.

First of all let us compute the Berezinian, i.e., the cohomology of complex (1) for $P = A$.

Theorem 2.9 [32].
(1) $\widehat{A}_n = 0$ *for* $n \neq s$.
(2) $\widehat{A}_s$ *is the module of sections for the bundle of volume forms* $\mathrm{Ber}(M)$ [3].

Proof. The assertion is local, so we can consider the domain $\mathcal{U}$ and split the complex $\mathrm{Diff}^+(\Lambda^*)$ in a tensor product of complexes $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}} \otimes \mathrm{Diff}^+(\Lambda^*)_{\mathrm{odd}}$, where $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}}$ is complex (1) on the underlying even domain of $\mathcal{U}$ and $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{odd}}$ is complex (1) for the Grassmann algebra in variables $\xi_1, \dots, \xi_t$.

It is known that $\mathrm{H}^i(\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}}) = 0$ for $i \neq s$ and $\mathrm{H}^i(\mathrm{Diff}^+(\Lambda^*)_{\mathrm{even}}) = \Lambda^s_u$, where Λ^s_u is the module of s-form on the underlying even domain of $\mathcal{U}$ (see [29, Section 2]). To compute the cohomology of $\mathrm{Diff}^+(\Lambda^*)_{\mathrm{odd}}$ consider the quotient complexes

$$0 \to \mathrm{Smbl}_k(A)_{\mathrm{odd}} \to \mathrm{Smbl}_{k+1}(\Lambda^1)_{\mathrm{odd}} \to \cdots,$$

where $\mathrm{Smbl}_k(P)_{\mathrm{odd}} = \mathrm{Diff}^+_k(P)_{\mathrm{odd}} / \mathrm{Diff}^+_{k-1}(P)_{\mathrm{odd}}$. An easy calculation shows that these complexes are the Koszul complexes, hence $\mathrm{H}^i(\mathrm{Diff}^+(\Lambda^*))_{\mathrm{odd}} = 0$ for $i > 0$ and $\mathrm{H}^0(\mathrm{Diff}^+(\Lambda^*))$ is a module of rank 1. Therefore $\widehat{A}_i = \mathrm{H}^i(\mathrm{Diff}^+(\Lambda^*)) = 0$ for $i \neq s$ and the only operators that represent non-trivial cocycles have the form $f(y, \xi)\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_s(\partial^t / \partial\xi_1 \cdots \partial\xi_t)$.

To complete the proof it remains to check that $\widehat{A}_s$ is precisely $\mathrm{Ber}(M)$, i.e., that changing coordinates we obtain

$$f \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_s \frac{\partial t}{\partial\xi_1 \cdots \xi_t} = f \,\mathrm{Ber}\, J\left(\frac{x}{z}\right) \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_s \frac{\partial t}{\partial\eta_1 \cdots \eta_t} + T,$$

where $z = (v_i, \eta_j)$ is a new coordinate system on $\mathcal{U}$, Ber denotes the Berezin determinant, $J(x/z)$ is the Jacobi matrix, T is cohomologous to zero. This is an immediate consequence of the following claim.

---

[3] Recall that locally sections of $\mathrm{Ber}(M)$ are written in the form $f(x)\mathbf{D}(x)$, where $f \in C^\infty(\mathcal{U})$ and $\mathbf{D}$ is a basis local section that is multiplied by the Berezin determinant of the Jacobi matrix under the change of coordinates. The Berezin determinant of an even matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is equal to $\det(A - BD^{-1}C)(\det D)^{-1}$.

Claim. *If* $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* $X^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ *are mutually inverse matrices written in the standard format, then* $\operatorname{Ber} X = \det A \cdot \det \tilde{D}$.

Proof. Obviously, $A\tilde{B} + B\tilde{D} = 0$ and $C\tilde{B} + D\tilde{D} = 1$, whence $D = \tilde{D}^{-1} + CA^{-1}B$. Therefore

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & \tilde{D}^{-1} + CA^{-1}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & \tilde{D}^{-1} \end{pmatrix}$$

and we get

$$\operatorname{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{Ber} \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \operatorname{Ber} \begin{pmatrix} A & B \\ 0 & \tilde{D}^{-1} \end{pmatrix} = \det A \cdot \det \tilde{D}. \qquad \square$$

Now let us consider the coordinate expression for adjoint operator. Suppose $\Delta \in \operatorname{Diff}_k(A, A)$ is a scalar operator:

$$\Delta = \sum_\sigma a_\sigma \frac{\partial^{|\sigma|}}{\partial x_\sigma},$$

where $\sigma = (i_1, \ldots, i_r)$ is an ordered set of integers $1 \le i_j \le s + t$, $|\sigma| = r$, $\partial^{|\sigma|}/\partial x_\sigma = \partial^{|\sigma|}/\partial x_{i_1} \cdots \partial x_{i_r}$. Then we have

$$\Delta^* = \sum_\sigma (-1)^\Theta \frac{\partial^{|\sigma|}}{\partial x_\sigma} \circ a_\sigma,$$

where $\Theta = |\sigma| + \widetilde{a_\sigma} \sum_{j=1}^{|\sigma|} \widetilde{x_{i_j}} + \sum_{1 \le j < k \le |\sigma|} \widetilde{x_{i_j}} \widetilde{x_{i_k}}$ and tilde over an object denotes the parity of the object. For a matrix d.o. $\Delta = \|\Delta_{ij}\|$ one has $(\Delta^*)_{ij} = (-1)^{\tilde{i}\tilde{j}} (\Delta_{ji})^*$.

3. Spencer complexes and Green's formula

3.1

From now onwards we assume that the module Λ^1 is of finite type and projective. This implies that the same is true for the modules Λ^k and $\mathcal{J}^k(A)$ and, also, that for any projective module the Spencer Diff-complex is exact (see [45]). In this case it is expedient to consider another variant of construction of adjoint operator.

Set $P^\circ = \operatorname{Hom}(P, B)$. Obviously, $\Sigma_i = \widehat{\Lambda^i} = (\Lambda^i)^\circ = \mathrm{D}_i(B)$, $\widehat{\mathcal{J}^k(A)} = \mathcal{J}^k(A)^\circ = \operatorname{Diff}_k(A, B)$.

Definition 3.1. For an operator $\Delta \in \operatorname{Diff}_k(A, B)$ we define the operator $\Delta^\circ : A \to B$, $\Delta^\circ(a) = \{\Delta, a\} j_k^*(a\Delta), a \in A$, which, for economy of language, will be called *adjoint*.

Proposition 3.2.

(1) *The grading of* Δ° *is equal to the one of* Δ.

(2) *If* $\Delta \in \mathrm{Diff}_k(A, B)$ *then* $\Delta^\circ \in \mathrm{Diff}_k(A, B)$.
(3) $\omega^\circ = \omega, \omega \in B = \mathrm{Diff}_0(A, B)$.
(4) *If* $X \in \mathrm{D}_1(B)$ *then* $X + X^\circ \in \mathrm{Diff}_0(A, B) = B$ *and* $X + X^\circ = \delta(X)$.
(5) $(a\Delta)^\circ = \{a, \Delta\}\Delta^\circ \circ a$, $\Delta \in \mathrm{Diff}(A, B)$, $a \in A$.
(6) *If* $\Delta(a) = [\nabla_a]$, $\Delta \in \mathrm{Diff}(A, B)$, $a \in A$, $\nabla_a \in \mathrm{Diff}^+(\Lambda^n)$, *then* $\Delta^\circ(a) = [\square_a]$, *where* $\square_a(a') = \{a, a'\}\nabla_{a'}(a)$.
(7) $(\Delta^\circ)^\circ = \Delta$.

Proof.
(1) Obvious.
(2) $\delta_a(\Delta^\circ)(a') = \{a, \Delta^\circ\}\Delta^\circ(aa') - a\Delta^\circ(a') = \{\Delta^\circ, a'\}j_k^*(aa'\Delta) - \{\Delta^\circ, a'\}aj_k^*(a'\Delta) = \{\Delta^\circ, a'\}\delta_a(j_k^*)(a'\Delta)$, so $\delta_{a_0,\dots,a_k}(\Delta^\circ)(a') = \{\Delta^\circ, a'\}\delta_{a_0,\dots,a_k}(j_k^*)(a'\Delta) = 0$.
(3) Obvious.
(4) Clearly, $\delta_a(j_1) = j_1(a) - aj_1(1) \in \mathcal{J}^1(A)$, hence

$$(\delta_a(j_1))^*(\Delta) = \{a, \Delta\}\Delta(a) - a\Delta(1) = \delta_a(\Delta)(1), \qquad \Delta \in \mathrm{Diff}_1(A, A).$$

Thus $\delta_a(X + X^\circ)(1) = \delta_a(X)(1) + \delta_a\left(j_1^*\right)(X) = \delta_a(X)(1) - \delta_a(j_1)^*(X) = 0$. Further, $\delta(X) = j_1^*(\gamma^*(X))$, where $\gamma : \mathcal{J}^1(A) \to \Lambda^1$ is the natural projection. Obviously, $\gamma^* : \mathrm{D}_1(B) \to \mathrm{Diff}_1(A, B)$ is the natural inclusion, so that $\delta(X) = X^\circ(1) = X + X^\circ$.
(5) $a\Delta^\circ\left(a'\right) = \{a\Delta, a'\}j_k^*\left(a'a\Delta\right) = \{a\Delta, a'\}\{a'a, \Delta\}\Delta^\circ\left(a'a\right) = \{a, \Delta\}\left(\Delta^\circ \circ a\right)\left(a'\right)$.
(6) Obvious.
(7) Let $\Delta(a) = [\nabla_a]$ and $\Delta^\circ(a) = [\square_a]$. Then $(\Delta^\circ)^\circ(a) = [\widetilde{\square_a}]$, where $\widetilde{\square_a}\left(a'\right) = \{a, a'\}\square_{a'}(a) = \nabla_a\left(a'\right)$, i.e., $(\Delta^\circ)^\circ(a) = [\nabla_a] = \Delta(a)$. □

Now let us define the adjoint operator Δ° in the general case when $\Delta \in \mathrm{Diff}(P, Q^\circ)$, the P and Q being A-modules. As in nongraded case consider the family of operators $\Delta(p, q) \in \mathrm{Diff}(A, B)$, $p \in P$, $q \in Q$, $\Delta(p, q)(a) = \{p, a\}\{q, a\}\langle\Delta(ap), q\rangle$, $a \in A$; the brackets $\langle\cdot,\cdot\rangle$ denote the natural pairing $Q^\circ \otimes Q \to B$. Then set $\langle\Delta^\circ(q), p\rangle = \{q, p\}\Delta(p, q)^\circ(1)$, $\Delta^\circ \in \mathrm{Diff}(Q, P^\circ)$.

Proposition 3.3.
(1) *For any* $\Delta \in \mathrm{Diff}_k(P, Q^\circ)$ *the operator* Δ° *is well-defined and of order* $\le k$.
(2) *For every* $\Delta \in \mathrm{Diff}(P, Q^\circ)$ *we have* $(\Delta^\circ)^\circ = \Delta$.
(3) *If the modules* P, Q, Q° *are projective, then for* $\Delta \in \mathrm{Diff}(P, Q^\circ)$ *the adjoint operator* $\Delta^\circ : Q \to P^\circ$ *coincides with the composition of* $Q \to Q^{\circ\circ} \xrightarrow{\Delta^*} P^\circ$, *where* $Q \to Q^{\circ\circ}$ *is the natural homomorphism.*

Proof. Statements (1) and (2) are straightforward.
(3) Take $\Delta : P \to Q^\circ$, $p \in P$, $q \in Q$. We have $\Delta = \psi_\Delta j_k$, so $\langle\Delta^*(q), p\rangle = \left\langle j_k^*\psi_\Delta^*(q), p\right\rangle = j_k^*\left(\psi_\Delta^*(q) \circ p\right) = \left(\psi_\Delta^*(q) \circ p\right)^\circ(1) = \{q, p\}\Delta(p, q)^\circ(1)$. □

Now let us consider one more example of adjoint operators.

3.2. *Example: The Spencer operators*

The complex

$$0 \to P \xrightarrow{j} \mathcal{J}^\infty(P) \xrightarrow{s} \mathcal{J}^\infty(P) \otimes \Lambda^1 \xrightarrow{s} \mathcal{J}^\infty(P) \otimes \Lambda^2 \xrightarrow{s} \cdots,$$

where $s(j(p) \otimes \omega) = j(p) \otimes d\omega$, is called the Spencer $\mathcal{J}$-complex (for details see [45]). Note that s is a d.o. of order ≤ 1. Let us compute the dual complex. Clearly, for the projective module P we have $\widehat{\mathcal{J}^\infty(P) \otimes \Lambda^i} = \left(\mathcal{J}^\infty(P) \otimes \Lambda^i\right)^\circ = \mathrm{D}_i(\mathrm{Hom}(\mathcal{J}^\infty(P), B)) = \mathrm{D}_i(\mathrm{Diff}(P, B)) = \mathrm{D}_i\left(\mathrm{Diff}^+(P^\circ)\right)$. To describe s^* we need the following remark.

Let $\Delta(P) : \mathrm{F}(P) \to \mathrm{G}(P)$ be a *natural* d.o., F and G be functors on a category of projective modules over A. For any functors F denote by $\dot{\mathrm{F}}(P)$ the abelian group $\mathrm{F}(P)$ supplied with A-module structure induced by the A-module structure in P. Then Δ generates the natural homomorphism $\dot{\Delta} : \dot{\mathrm{F}} \to \dot{\mathrm{G}}$.

Lemma 3.4. $(\Delta^*)^{\cdot} = \left(\dot{\Delta}\right)^*$

Proof. It is trivial. □

The lemma immediately implies that the complex dual to the Spencer $\mathcal{J}$-complex for module P is the Spencer Diff-complex for module P°. On the other hand we have $\widehat{\mathcal{J}^\infty(P) \otimes \Lambda^k} = \mathrm{Hom}(\mathcal{J}^\infty(P), \mathrm{D}_k(B)) = \mathrm{Diff}(P, \Sigma_k)$ and the operator $s^* : \mathrm{Diff}(P, \Sigma_k) \to \mathrm{Diff}(P, \Sigma_{k-1})$ has the form $s^*(\nabla) = \delta \circ \nabla$, $\nabla \in \mathrm{Diff}(P, \Sigma_k)$; $j^*(\nabla) = \nabla^\circ(1)$. Bringing all this together we can state the following theorem.

Theorem 3.5. *For a projective module P there is an isomorphism of complexes*

$$\begin{tikzcd}
0 & P^\circ \arrow[l] \arrow[d, equal] & \mathrm{Diff}^+(P^\circ) \arrow[l, "\mathcal{D}"'] \arrow[d, "\psi"'] & \mathrm{D}_1\left(\mathrm{Diff}^+(P^\circ)\right) \arrow[l, "S"'] \arrow[d, "\psi"'] & \mathrm{D}_2\left(\mathrm{Diff}^+(P^\circ)\right) \arrow[l, "S"'] \arrow[d, "\psi"'] & \cdots \arrow[l] \\
0 & P^\circ \arrow[l] & \mathrm{Diff}(P, B) \arrow[l, "\mu"'] & \mathrm{Diff}(P, \Sigma_1) \arrow[l, "\omega"'] & \mathrm{Diff}(P, \Sigma_2) \arrow[l, "\omega"'] & \cdots \arrow[l]
\end{tikzcd}$$

where $\omega(\nabla) = \delta \circ \nabla$, $\nabla \in \mathrm{Diff}(P, \Sigma_k)$, $\mu(\nabla) = \nabla^\circ(1)$, *the maps ψ are the composition of isomorphisms*:

$$\mathrm{D}_k\left(\mathrm{Diff}^+(P^\circ)\right) \xrightarrow{\mathrm{D}_k(\circ)} \mathrm{D}_k(\mathrm{Diff}(P, B)) \to \mathrm{Diff}(P, \mathrm{D}_k(B)) \to \mathrm{Diff}(P, \Sigma_k).$$

This theorem has the following corollary.

Corollary 3.6 (Green's formula). *If the Spencer* Diff*-cohomology of the Berezinian B in the term* $\mathrm{Diff}^+(B)$ *is trivial, then for any $\Delta \in \mathrm{Diff}(P, Q^\circ)$, $p \in P$, $q \in Q$, we have*

$$\langle \Delta(p), q \rangle - \{\Delta, p\}\left(p, \Delta^\circ(q)\right) = \delta G,$$

where $G \in \Sigma_1$ is an integral 1-form.

Proof. Suppose $\nabla \in \mathrm{Diff}^+(B)$; then $(\nabla - \nabla(1)) \in \ker \mathcal{D}$, hence there exists $\Box \in \mathrm{D}_1(\mathrm{Diff}^+(B))$ such that $S(\Box) = \nabla - \nabla(1)$. It follows from the theorem that $\nabla^*(1) - \nabla(1) = \psi \circ S(\Box) = \omega(\psi(\Box))$. Therefore $\nabla^*(1) - \nabla(1) = \delta G$, where $G = \psi(\Box)(1)$. Letting $\nabla = \Delta(p,q)$, we get the Green formula. □

Remark 3.7. If B is projective then there exists an A-homomorphism $\varkappa : \ker \mathcal{D} \to \mathrm{D}_1(\mathrm{Diff}^+(B))$, such that $S \circ \varkappa = \mathrm{id}$, and we can take $\Box = \varkappa(\Delta - \Delta(1))$ and, therefore, $G = G_\varkappa = \psi(\varkappa(\Delta(p,q) - \langle \Delta(p), q\rangle))(1)$. Since the Spencer complex of B is exact, for any two homomorphisms $\varkappa$ and $\varkappa'$ one can find a homomorphism $f : \ker \mathcal{D} \to \mathrm{D}_2(\mathrm{Diff}^+(B))$ such that $\varkappa - \varkappa' = S \circ f$. Hence $G_\varkappa - G_{\varkappa'} = \delta F$, where $F = \psi(f(\Delta(p,q) - \langle \Delta(p), q\rangle))(1)$.

3.3

In this subsection we describe a spectral sequence, which establishes the relationship between the de Rham cohomology and the homology of the complex of integral forms.

Proposition 3.8 (Poincaré duality). *There is a spectral sequence, $(E^r_{*,*}, d^r_*)$, with*

$$E^2_{p,q} = \mathrm{H}_p\left((\Sigma_*)_{-q}\right),$$

the homology of complexes of integral forms, and converging to the de Rham cohomology $\mathrm{H}(\Lambda^*)$.

Proof. Let $K_{p,q} = \mathrm{D}_p\left(\mathrm{Diff}^+(\Lambda^{-q})\right)$, d' be the Spencer operator $d' : K_{p,q} \to K_{p-1,q}$, and $d'' = (-1)^p \mathrm{D}_p\left(\mathrm{Diff}^+(d)\right)$, $d'' : K_{p,q} \to K_{p,q-1}$. Then $\{K_{*,*}, d', d''\}$ is a double complex. Obviously, ${}^{\mathrm{I}}\mathrm{H}(K) = \mathrm{H}\left(K_{*,*}, d'\right) = \Lambda^{-q}$ for $p = 0$ and 0 for $p \neq 0$. Therefore in the second spectral sequence ${}^{\mathrm{II}}E^2_{p,q} = {}^{\mathrm{II}}\mathrm{H}_{p,q}\left({}^{\mathrm{I}}\mathrm{H}(K)\right) = \mathrm{H}_{p,q}\left({}^{\mathrm{I}}\mathrm{H}_{*,*}(K), d''\right) = \mathrm{H}^{-q}(\Lambda^*)$ for $p = 0$ and ${}^{\mathrm{II}}E^2_{p,q} = 0$ for $p \neq 0$. Thus the second spectral sequence converge to the de Rham cohomology. From the first spectral sequence we get ${}^{\mathrm{I}}E^2_{p,q} = {}^{\mathrm{I}}\mathrm{H}_{p,q}\left({}^{\mathrm{II}}\mathrm{H}(K)\right) = {}^{\mathrm{I}}\mathrm{H}_{p,q}\left(\mathrm{H}\left(K_{*,*}, d''\right)\right) = \mathrm{H}_{p,q}\left(\mathrm{D}_*(B), d'\right) = \mathrm{H}_p\left((\Sigma_*)_{-q}\right)$. □

Corollary 3.9. *If $\widehat{A_i} = 0$ for all $i \neq n$, then* $\mathrm{H}_i(\Sigma_*) = \mathrm{H}^{n-i}(\Lambda^*)$.

4. Quadratic Lagrangians, the Euler operator, and the Noether theorem

In this section P and Q are projective modules.

4.1

Consider the complex

$$0 \leftarrow \mathrm{Diff}^{\mathrm{sym}}\left(P, P^\circ\right) \xleftarrow{\mu} \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, B) \xleftarrow{\omega} \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, \Sigma_1) \xleftarrow{\omega} \cdots, \tag{2}$$

where $\mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, Q)$ denotes the submodule of $\mathrm{Diff}\,(P, \mathrm{Diff}\,(P, Q))$ consisting of all symmetric bidifferential Q-valued operators $\nabla(p_1)(p_2) = \{p_1, p_2\}\nabla(p_2)(p_1)$, $\nabla \in \mathrm{Diff}\,(P, \mathrm{Diff}\,(P, Q))$, $\mathrm{Diff}^{\mathrm{sym}}(P, P^\circ)$ is the module of all self-adjoint operators from $\mathrm{Diff}\,(P, P^\circ)$, $\omega(\nabla) = \delta \circ \nabla$, $\nabla \in \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, \Sigma_k)$, $k > 0$, and $\mu(\nabla)(p) = (\nabla(p))^\circ(1)$, $p \in P$.

Theorem 4.1. *This complex is acyclic.*

Proof. From Theorem 3.5 it follows easily that the complex

$$0 \leftarrow \mathrm{Diff}\left(P, P^\circ\right) \xleftarrow{\tilde{\mu}} \mathrm{Diff}\,(P, \mathrm{Diff}\,(P, B)) \xleftarrow{\tilde{\omega}} \mathrm{Diff}\,(P, \mathrm{Diff}\,(P, \Sigma_1)) \xleftarrow{\tilde{\omega}} \cdots, \quad (3)$$

where $\tilde{\omega}(\nabla) = \delta \circ \nabla$, $\tilde{\mu}(\nabla)(p) = (\nabla(p))^\circ(1)$, is acyclic. To prove the theorem it is sufficient to show that this complex splits into the symmetric and the skew-symmetric parts. To do this let us check that the involution ρ of complex (3), $\rho(\nabla)(p_1)(p_2) = \{p_1, p_2\}\nabla(p_2)(p_1)$, $\nabla \in \mathrm{Diff}\,(P, \mathrm{Diff}\,(P, \Sigma_k))$ and $\rho(\nabla) = \nabla^\circ$, $\nabla \in \mathrm{Diff}\,(P, P^\circ)$, is an automorphism of this complex. The fact that $\tilde{\omega} \circ \rho = \rho \circ \tilde{\omega}$ is obvious. Let us verify that $\tilde{\mu} \circ \rho = \rho \circ \tilde{\mu}$. Take $\Delta \in \mathrm{Diff}\,(P, \mathrm{Diff}\,(P, B))$ and let $\delta(p_1)(p_2) = [\nabla_{p_1, p_2}]$, $\nabla_{p_1, p_2} \in \mathrm{Diff}^+(\Lambda^n)$. It follows from Proposition 3.2 that $\langle \tilde{\mu}(\Delta)(p_1), p_2\rangle = [\square]$, where $\square \in \mathrm{Diff}^+(\Lambda^n)$, $\square(a) = \{p_2, a\}\nabla_{p_1, ap_2}(1)$, $p_1, p_2 \in P$. Therefore $\langle \rho\tilde{\mu}(\Delta)(p_1), p_2\rangle = \langle \tilde{\mu}(\Delta)^\circ(p_1), p_2\rangle = \{p_1, p_2\}[\square']$, where $\square'(a) = \{p_1, a\}\{p_2, a\}\nabla_{ap_2, p_1}(1)$. On the other hand $\langle \tilde{\mu}\rho(\Delta)(p_1), p_2\rangle = [\square'']$, where $\square''(a) = \{p_2, a\}\nabla'_{p_1, ap_2}(1)$, $\langle \rho(\Delta)(p_1), p_2\rangle = [\nabla'_{p_1, p_2}]$. Clearly, $\nabla'_{p_1, p_2} = \{p_1, p_2\}\nabla_{p_2, p_1}$, hence $\square''(a) = \{p_2, a\}\{p_1, ap_2\}\nabla_{ap_2, p_1}(1)$. □

4.2. Lagrangian formalism

Definition 4.2. The space $\mathcal{L}ag\,(P)$ of *quadratic Lagrangians* on P is defined as the cokernel of the operator $\omega : \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, \Sigma_1) \to \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, B)$. An operator $L \in \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, B)$ is called the *density* of quadratic Lagrangian $\mathcal{L}$ if $\mathcal{L} = L \bmod \mathrm{im}\,\omega$.

From Theorem 4.1 we see that the operator μ gives rise to an isomorphism of $\mathcal{L}ag\,(P)$ to the submodule of the module $\mathrm{Diff}\,(P, P^\circ)$ consisting of self-adjoint operators. This isomorphism is said to be the *Euler operator* and denoted by $\mathcal{E}$.

4.3. Conservation laws

Let $\Delta \in \mathrm{Diff}_k(P, Q)$ and $E = \{p \in P \mid \Delta(p) = 0\}$ is the corresponding equation. The operator Δ generates the chain map Ω_Δ of the complexes (2):

$$\begin{array}{ccccccccc}
0 \longleftarrow & \mathrm{Diff}\,(Q, B) & \xleftarrow{\omega} & \mathrm{Diff}\,(Q, \Sigma_1) & \xleftarrow{\omega} & \mathrm{Diff}\,(Q, \Sigma_2) & \xleftarrow{\omega} \cdots \\
 & \downarrow{\scriptstyle \Omega_\Delta} & & \downarrow{\scriptstyle \Omega_\Delta} & & \downarrow{\scriptstyle \Omega_\Delta} & \\
0 \longleftarrow & \mathrm{Diff}\,(P, B) & \xleftarrow{\omega} & \mathrm{Diff}\,(P, \Sigma_1) & \xleftarrow{\omega} & \mathrm{Diff}\,(P, \Sigma_2) & \xleftarrow{\omega} \cdots
\end{array}$$

(Linear) conservation laws for the equation E are defined by analogy with nongraded case (see [29]) as classes of 1-dimensional homology of the complex coker Ω_Δ. Let us denote the group of linear conservation laws for the equation $\Delta = 0$ by $\mathrm{Cl}(\Delta) = \mathrm{H}_1(\operatorname{coker} \Omega_\Delta)$. The following theorem and the corollary describe the group $\mathrm{Cl}(\Delta)$.

Theorem 4.3. *There exists an exact sequence*

$$0 \to \mathrm{H}_1(\operatorname{im} \Omega_\Delta) \to \mathrm{H}_0(\ker \Omega_\Delta) \to \ker \Delta^* \to \mathrm{Cl}(\Delta) \to 0.$$

Proof. It follows from Theorem 4.1 that exact homology sequences corresponding to the short exact sequences of complexes

$$0 \to \ker \Omega_\Delta \to \mathrm{Diff}(Q, \Sigma_*) \to \operatorname{im} \Omega_\Delta \to 0$$

$$0 \to \operatorname{im} \Omega_\Delta \to \mathrm{Diff}(p, \Sigma_*) \to \operatorname{coker} \Omega_\Delta \to 0$$

have the form

$$0 \to \mathrm{H}_1(\operatorname{im} \Omega_\Delta) \to \mathrm{H}_0(\ker \Omega_\Delta) \xrightarrow{i_1} Q^\circ \xrightarrow{i} \mathrm{H}_0(\operatorname{im} \Omega_\Delta) \to 0$$

$$0 \to \mathrm{H}_1(\operatorname{coker} \Omega_\Delta) \to \mathrm{H}_0(\operatorname{im} \Omega_\Delta) \xrightarrow{j} P^\circ \to \mathrm{H}_0(\operatorname{coker} \Omega_\Delta) \to 0.$$

It is straightforward to check that the composition $j \circ i : Q^\circ \to P^\circ$ coincides with the adjoint operator $\Delta^* : Q^\circ \to P^\circ$. Hence $\ker \Delta^* / \operatorname{im} i_1$ is isomorphic to $\ker j = \mathrm{Cl}(\Delta)$ and we get the desired exact sequence. □

Corollary 4.4. *If* $\ker \Omega_\Delta = 0$ *then the group of linear conservation laws* $\mathrm{Cl}(\Delta)$ *is isomorphic to* $\ker \Delta^*$.

Let us give an explicit expression for the map $\ker \Delta^* \to \mathrm{Cl}(\Delta)$. Suppose $q^\circ \in \ker \Delta^* \subset Q^\circ$. Then, choosing a homomorphism $\varkappa$ (see Remark 3.7), we have a d.o. from P to Σ_1, $p \mapsto G_\varkappa(\Delta(p, q^\circ))$. The Green formula yields $\langle \Delta(p), q^\circ \rangle = \delta G_\varkappa(\Delta(p, q^\circ))$. Hence the operator $p \mapsto G_\varkappa(\Delta(p, q^\circ))$ is a 1-cocycle of the complex coker Ω_Δ and we obtain a map $\chi : \ker \Delta^* \to \mathrm{Cl}(\Delta)$, where $\chi(q^\circ)$ is the conservation law corresponding to the operator $p \mapsto G_\varkappa(\Delta(p, q^\circ))$. Let us show that this is the map under consideration. If ∇ is a 1-cocycle of coker Ω_Δ, the element q° from $\ker \Delta^*$ corresponding to it according to the proof of Theorem 4.3 can be found as $q^\circ = \mu(\Box)$, where $\Box \in \mathrm{Diff}(Q, B)$ satisfy the relation $\Box \circ \Delta = \delta \circ \nabla$. If ∇ is the operator $p \mapsto G_\varkappa(\Delta(p, q^\circ))$ then $\Box = q^\circ \in \mathrm{Diff}(P, B)$ and $\mu(\Box) = \mu(q^\circ) = q^\circ$.

4.4. The Noether theorem

We start with a description of transformations of the objects that Noether's theorem includes.

Let $\mathrm{Der}(P) = \{(X_P, X) \mid X \in \mathrm{D}_1(A), X_P \in \mathrm{Diff}_1(P, P), \text{and } \forall a \in A, \forall p \in P$ $X_P(ap) = \{X_P, a\} a X_P(p) + X(a)p\}$. If $X \in \mathrm{D}_1(A)$ then $(X_B, X) \in \mathrm{Der}(B)$, where

$X_B = -X^*$ (more generally, if $(X_P, X) \in \mathrm{Der}(P)$ then one can define $(X_{P^\circ}, X) \in \mathrm{Der}(P^\circ)$, $X_{P^\circ} = -(X_P)^*$).

Given $(X_P, X) \in \mathrm{Der}(P)$, define $(X_{\mathrm{Diff}}, X) \in \mathrm{Der}(\mathrm{Diff}(P, B))$ by the formula $X_{\mathrm{Diff}}(\Delta) = [X, \Delta] = X_B \circ \Delta - \{X, \Delta\}\Delta \circ X_P$. For $L \in \mathrm{Diff}(P, \mathrm{Diff}(P, B))$ we put $X_P(L) = X_{\mathrm{Diff}} \circ L - \{X, L\}L \circ X_P$. If $L \in \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, B)$ then $X_P(L) \in \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, B)$ and is called the *variation* of L under the infinitesimal transformation X_P. Clearly, X_P generates the map of Lagrangians on P, $X_P : \mathcal{L}ag(P) \to \mathcal{L}ag(P)$.

From the definition of variation it follows that

$$X_P(L)(p_1)(p_2) = X_B(L(p_1)(p_2)) - \{X, L\}\{X, p_1\}L(p_1)(X_P(p_2)) \\ -\{X, L\}L(X_P(p_1))(p_2).$$

Further, using Proposition 3.2 (4)we get

$$X_B(L(p_1)(p_2)) = -\{X, L\}\{X, p_1\}\{X, p_2\}\delta(L(p_1)(p_2) \circ X).$$

It follows from Green's formula that $\forall p_1, p_2 \in P$

$$L(p_1, p_2) = \langle \mathcal{E}_L(p_1), p_2\rangle + \delta G_\varkappa(L(p_1)(p_2, 1)).$$

Combining these formulae, we obtain the following *formula for the first variation*:

$$X_P(L)(p_1)(p_2) = -\{L, p_2\}\{p_1, p_2\}\langle X_P(p_2), \mathcal{E}_L(p_1)\rangle \\ -\{L, p_1\}\langle X_P(p_1), \mathcal{E}(p_2)\rangle - \delta n_\varkappa(p_1, p_2),$$

where

$$\begin{aligned} n_\varkappa(p_1, p_2) = {} & \{X, L\}\{X, p_1\}\{X, p_2\}L(p_1)(p_2) \circ X \\ & +\{X, L\}\{X, p_1\}G_\varkappa(p_1, X_P(p_2)) \\ & + \{X, L\}\{X, p_2\}\{p_1, p_2\}G_\varkappa(p_2, X_P(p_1)). \end{aligned}$$

Definition 4.5. $X_P \in \mathrm{Der}(P)$ is said to be a *symmetry* of a Lagrangian $\mathcal{L}$ if $X_P(\mathcal{L}) = 0$.

From Theorem 4.1 it follows that a symmetry of $\mathcal{L}$ is a symmetry of the operator $\mathcal{E}_\mathcal{L} = \mathcal{E}(\mathcal{L})$, i.e., $X_P(\mathcal{E}_\mathcal{L}) = X_{P^\circ} \circ \mathcal{E}_\mathcal{L} - \{X, \mathcal{E}_\mathcal{L}\}\mathcal{E}_\mathcal{L} \circ X_P = 0$, and conversely.

If X_P is a symmetry of $\mathcal{L}$, then $X_P(L) = \omega(L')$, where L is a density of $\mathcal{L}$, $L' \in \mathrm{Diff}^{\mathrm{sym}}_{(2)}(P, \Sigma_1)$. Consider the integral 1-form $n_\varkappa(p_1, p_2) + L'(p_1)(p_2)$. The first variation formula implies that this integral form is closed whenever $p_1, p_2 \in \ker \mathcal{E}_\mathcal{L}$. Clearly, its homological class does not depend on the choice of $\varkappa$ and L'. Thus we proved the following theorem.

Theorem 4.6 (Noether). *If X_P is a symmetry of the Lagrangian $\mathcal{L} = L \bmod \operatorname{im} \omega$ and the module P is projective, then the map $p \mapsto n_\varkappa(p, p) + L'(p)(p)$, $p \in P$, gives rise to a conservation law of the equation $\mathcal{E}_\mathcal{L} = 0$.*

Appendix A. Right connections

This appendix provides a lightening sketch of the algebraic formalism of right connections.

Remark. This issue is also treated in [53], the approach, although different, being close to the approach suggested here.

A.1.

We begin with a collection of a few facts about (usual) linear connections.

One can consider a *connection* on an A-module P as an A-homomorphism of grading zero, $\gamma : P \mapsto \mathcal{J}^1(P)$, satisfying $\nu \circ \gamma = \mathrm{id}_P$, where $\nu : \mathcal{J}^1(P) \to P$ is the natural projection.

The composition $\Lambda^i \otimes P \xrightarrow{\mathrm{id}\otimes\gamma} \Lambda^i \otimes \mathcal{J}^1(P) \xrightarrow{s} \Lambda^{i+1} \otimes P$ of γ and the Spencer operator s is called the *de Rham operator* associated with γ and denoted by $d : \Lambda^i \otimes P \to \Lambda^{i+1} \otimes P$. The first de Rham operator $d : P \to \Lambda^1 \otimes P$ gives rise, in a natural way, to a *covariant derivative*, i.e., a homomorphism $\nabla : \mathrm{D}_1(A) \to \mathrm{Diff}_1(P, P)$, $X \mapsto \nabla_X$, such that $\nabla_X(ap) = \{X, a\}a\nabla_X(p) + X(a)\,p$, $X \in \mathrm{D}_1(A)$. The de Rham sequence with values in P,

$$0 \to P \xrightarrow{d} \Lambda^1 \otimes P \xrightarrow{d} \Lambda^2 \otimes P \xrightarrow{d} \cdots,$$

is not a complex in general. The operator $d^2 : \Lambda^i \otimes P \to \Lambda^{i+2} \otimes P$ is said to be *curvature* of the connection γ. It is straightforward to check that d^2 is a Λ^*-linear operator and, therefore, for P projective the curvature d^2 is multiplication by a $R_\gamma \in \Lambda^2 \otimes \mathrm{Hom}_A(P, P)$.

A.2.

We define a right connection in a dual way.

Definition. A *right connection* on P is defined to be an A-homomorphism of grading zero $\lambda : \mathrm{Diff}_1^+(P) \to P$ satisfying $\lambda \circ \iota = \mathrm{id}_P$, where $\iota : P \to \mathrm{Diff}_1^+(P)$ is the natural inclusion.

Given a module P with a right connection λ, one can carry out the construction of the *sequence of integral form with values in P* by letting the operator $\delta : \mathrm{D}_{i+1}(P) \to \mathrm{D}_i(P)$ be the composition $\mathrm{D}_{i+1}(P) \xrightarrow{S} \mathrm{D}_i\left(\mathrm{Diff}_1^+(P)\right) \xrightarrow{\mathrm{D}_i(\lambda)} \mathrm{D}_i(P)$ of the Spencer operator S and $\mathrm{D}_i(\lambda)$:

$$0 \leftarrow P \xleftarrow{\delta} \mathrm{D}_1(P) \xleftarrow{\delta} \mathrm{D}_2(P) \xleftarrow{\delta} \cdots$$

The first operator $\delta : \mathrm{D}_1(P) \to P$ provides a *right covariant derivative*, i.e., a homomorphism $\nabla : \mathrm{D}_1(A) \to \mathrm{Diff}_1^+(P, P)$, $X \mapsto \nabla_X$, such that

$$\nabla_X (ap) = \{X, a\} a \nabla_X (p) - X(a)\, p, \qquad X \in \mathrm{D}_1(A). \tag{A.1}$$

Remark. This "right Leibniz rule" may be reformulated by the following way. The operator ∇ can be extended to an A-homomorphism $\nabla : \mathrm{Diff}_1(A, A) \to \mathrm{Diff}_1^+(P, P)$ by putting $\nabla_{\mathrm{id}_A} = \mathrm{id}_P$. Then (1) means that the map $\nabla : \mathrm{Diff}_1^+(A, A) \to \mathrm{Diff}_1(P, P)$ is also an A-homomorphism.

The operator $\delta^2 : \mathrm{D}_{i+2}(P) \to \mathrm{D}_i(P)$ is called the *curvature* of the right connection λ. A direct calculation shows that this is a $\mathrm{D}_*(A)$-linear operator and, so, for P projective the curvature δ^2 can be interpreted as inner product with a $R_\lambda \in \Lambda^2 \otimes \mathrm{Hom}_A(P, P)$.

Example. If on a projective module P there is a connection $\gamma : P \to \mathcal{J}^1(P)$, then on P° there is a right connection $\lambda = \gamma^* : (\mathcal{J}^1(P))^\circ = \mathrm{Diff}_1^+(P^\circ) \to P^\circ$. In particular, the obvious flat connection on $A, \gamma : A \to \mathcal{J}^1(A)$, $\gamma(a) = a j_1(1)$, provides the canonical flat right connection on the Berezinian B. The complex of integral forms with values in P° is dual to the de Rham complex with coefficients in P.

Acknowledgements

The author is grateful to Professor A.M. Vinogradov for very stimulating attention to this work and to I.S. Krasil'shchik, M.M. Vinogradov, and D.M. Guessler for useful discussions and their remarks on the preprint of this work.

He also wishes to thank the SISSA for warm hospitality and pleasant atmosphere in which this work was done and the Italian Ministero degli Affari Esteri for a fellowship.

It is a great pleasure for the author to give his special thanks to the scientific attaché at the Italian Embassy in Moscow Professor G. Piragino for an understanding and help which made possible his stay at the SISSA.

Finally, many thanks to the referee who indicated some of the relevant references added in the text.

References

[1] D.V. Alekseevskij, A.M. Vinogradov and V.V. Lychagin, Basic ideas and concepts of differential geometry, Itogi nauki i tekhniki, Sovremennye problemy matematiki, Fundamental'nye napravleniya, Tom 28, Geometriya I, VINITI, Moscow (1988) (Russian); English transl. Encyclopaedia of mathematical sciences, Vol. 28 Geometry I (Springer, Berlin and Heidelberg, 1991).

[2] Yu.I. Manin, *Topics in Noncommutative Geometry* (Princeton University Press, Princeton, 1991).

[3] D. Hernández Ruipérez and J. Muñoz Masqué, Global variational calculus on graded manifolds, I: Graded jet bundles, structure 1-form and graded infinitesimal contact transformations, J. Math. Pures Appl. 63 (1984) 283–309.

[4] D. Hernández Ruipérez and J. Muñoz Masqué, Global variational calculus on graded manifolds, II J. Math. Pures Appl. 64 (1985) 87–104.

[5] D. Hernández Ruipérez and J. Muñoz Masqué, Infinitesimal functoriality of graded Poincaré–Cartan forms, in: *Differential Geometric Methods in Theoretical Physics,* Proc. XIII Int. Conf. in Shumen (Bulgaria), eds. H.-D. Doebner and T.D. Palev (World Scientific, Singapore, 1986) pp.126–132.

[6] B.A. Kupershmidt, An algebraic model of graded calculus of variations, Math. Proc. Cambridge Philos. Soc. 101 (1987) 151–166.

[7] B.A. Kupershmidt, *The Variational Principles of Dynamics* (World Scientific, Singapore, 1992).

[8] E.D. van der Lende, Super integrable systems, Ph.D. Thesis, University of Amsterdam (1991).

[9] J. Monterde, Higher order graded and Berezinian Lagrangian densities and their Euler–Lagrange equations, Ann. Inst. H. Poincaré. Phys. Théor. 57 (1992) 3–26.

[10] J. Monterde and J. Muñoz Masqué, Variational problems on graded manifolds, Contemp. Math. 132, Mathematical Aspects of Classical Field Theory, Amer. Math. Soc., Providence, RI (1992) 551–571.

[11] L.A. Ibort and J. Marín-Solano, Geometrical foundations of Lagrangian supermechanics and supersymmetry, Rep. Math. Phys. 32 (1993) 385–409.

[12] L.A. Ibort, G. Landi, J. Marín-Solano and G. Marmo, On the inverse problem of Lagrangian supermechanics, Internat. J. Modern Phys. A 8 (1993) 3565–3576.

[13] G.H.M. Roelofs, Prolongation structures of supersymmetric systems, Ph.D. Thesis, University of Twente, The Netherlands (1993).

[14] J. Rembieliński, Differential and integral calculus on the quantum $\mathbb{C}$-plane, in: *Quantum Groups and Related, Topics*, eds. R. Gielerak, J. Lukierski and Z. Popowicz (Kluwer Academic Publishers, Dordrecht, 1992) pp. 129–139.

[15] C. Chryssomalakos and B. Zumino, Translations, integrals and Fourier transformations in the quantum plane, preprint LBL-34803, UCB-PTH-93/30 (1993).

[16] A. Kempf and S. Majid, Algebraic q-integration and Fourier theory on quantum and braided spaces, J. Math. Phys. 35 (1994) 6802–6837.

[17] J.F. Cariñena and H. Figueroa, A geometrical version of Noether's theorem in supermechanics, preprint DFTUZ/94/09 (1994).

[18] J. Lukierski and V. Rittenberg, Color–de Sitter and color-conformal superalgebras, Phys. Rev. D (3) 18 (1978) 385–389.

[19] V. Rittenberg and D. Wyler, Generalized superalgebras, Nuclear Phys. B 139 (1978) 189–202.

[20] A. Borowiec, W. Marcinek and Z. Oziewicz, On multigraded differential calculus, in: *Quantum Groups and Related Topics*, eds. R. Gielerak, J. Lukierski and Z. Popowicz (Kluwer Academic Publishers, Dordrecht, 1992) pp. 103–114.

[21] M.A. Vasiliev, de Sitter supergravity with positive cosmological constant and generalized Lie superalgebras, Classical Quantum Gravity 2 (1985) 645–659.

[22] M. Omote, Y. Ohnuki and S. Kamefuchi, Fermi–Bose similarity, Progr. Theoret. Phys. 56 (1976) 1948–1964.

[23] M. Omote and S. Kamefuchi, On supergroup transformations, Nuovo Cimento A (11) 50 (1979) 21–40.

[24] Y. Ohnuki and S. Kamefuchi, Fermi–Bose similarity, supersymmetry and generalized numbers, Nuovo Cimento A (11) 70 (1982) 435–459.

[25] Y. Ohnuki and S. Kamefuchi, Fermi–Bose similarity, supersymmetry and generalized numbers - II, Nuovo Cimento A (11) 77 (1983) 99–119.

[26] R. Matthes, "Quantum group" structure and "covariant" differential calculus on symmetric algebras corresponding to commutation factors on $\mathbb{Z}^n$, in: *Quantum Groups and Related Topics*, eds. R. Gielerak, J. Lukierski and Z. Popowicz (Kluwer Academic Publishers, Dordrecht, 1992) pp. 45–54.

[27] P.J.M. Bongaarts and H.G.J. Pijls, Almost commutative algebra and differential calculus on the quantum hyperplane, J. Math. Phys. 35 (1994) 959–970.

[28] A.M. Vinogradov, A spectral sequence associated with a nonlinear differential equation and algebro-geometric foundations of Lagrangian field theory with constraints, Dokl. Akad. Nauk SSSR 238 (1978) 1028–1031 (Russian); English transl. in Soviet Math. Dokl. 19 (1978) 144–148.

[29] A.M. Vinogradov, The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws, I. The linear theory, II. The nonlinear theory, J. Math. Anal. Appl. 100 (1984) 1–129.

[30] A.M. Vinogradov, From symmetries of partial differential equations towards secondary ("quantized") calculus, J. Geom. Phys. 14 (1994) 146–194.

[31] I.M. Anderson, Introduction to the variational bicomplex, Contemp. Math. 132, Mathematical Aspects of Classical Field Theory, Amer. Math. Soc., Providence, RI (1992) 51–73.

[32] I.B. Penkov, $\mathcal{D}$-modules on supermanifolds, Invent. Math. 71 (1983) 501–512.

[33] A.M. Vinogradov, On the algebro-geometric foundations of Lagrangian field theory, Dokl. Akad. Nauk SSSR 236 (1977) 284–287 (Russian); English transl. in Soviet Math. Dokl. 18 (1977) 1200–1204.

[34] Yu.I. Manin and I.B. Penkov, The formalism of left and right connections on supermanifolds, *Lectures on Supermanifolds, Geometrical Methods and Conformal Groups*, eds. H.-D. Doebner, J.D. Henning, and T.D. Palev (World Scientific Publishing, Teaneck, NJ, 1989) pp. 3–13.
[35] Yu.I. Manin, Gauge field theory and complex geometry, Nauka, Moscow (1984) (Russian); English transl. (Springer, Berlin and Heidelberg, 1988).
[36] N. Bourbaki, *Éléments de mathématique, Algèbre*, Chapitres 1 à 3, Nouvelle édition, Hermann Paris (1970) (French); English transl. *Elements of Mathematics, Algebra*, Chapters 1–3, Hermann Paris (1974).
[37] Yu.I. Manin, *Quantum Groups and Non-commutative Geometry*, Lecture notes (CRM, Université de Montréal, 1989).
[38] M.A. Rieffel, Non-commutative tori—A case study of non-commutative differentiable manifolds, Contemp. Math. 105, Geometric and Topological Invariants of Elliptic Operators, Amer. Math. Soc., Providence, RI (1990) 191–211.
[39] V.V. Lychagin, *Quantum Mathematics*, Lectures given at the Moscow State University in 1992-93.
[40] V.V. Lychagin, Calculus and quantizations over Hopf algebras, preprint hep-th/9406097 (1994).
[41] V.V. Lychagin, Quantizations of braided differential operators, preprint ESI 51 (1993).
[42] V.V. Lychagin, Braided differential operators and quantization in ABC-categories, C. R. Acad. Sci. Paris Sér. I Math. 318 (1994) 857–862.
[43] A.M. Vinogradov, The logic algebra for the theory of linear differential operators, Dokl. Akad. Nauk SSSR 205 (1972) 1025–1028 (Russian); English transl. in Soviet Math. Dokl. 13 (1972) 1058–1062.
[44] A.M. Vinogradov, Geometry of nonlinear differential equations, Itogi nauki i tekhniki, Problemy geometrii, Tom 11 (1980) 89–134 (Russian); English transl. in J. Soviet Math. 17 (1981) 1624–1649.
[45] A.M. Vinogradov, I.S. Krasil'shchik, and V.V. Lychagin, *Introduction to the Geometry of Nonlinear Differential Equations*, Nauka, Moscow (1986) (Russian); English transl. *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, New York, 1986).
[46] M.M. Vinogradov, Fundamental functors of differential calculus in graded algebras, Uspekhi Mat. Nauk 44 (1989) 151–152 (Russian); English transl. in Russian Math. Surveys 44 (1989) 220–221.
[47] D.A. Leites, Introduction to the theory of supermanifolds, Uspekhi Mat. Nauk 35 (1980) 3–57 (Russian); English transl. in Russian Math. Surveys 35 (1980) 1–64.
[48] I.N. Bernshtein and D.A. Leites, Integral forms and the Stokes formula on supermanifolds, Funktsional. Anal. i Prilozhen. 11 (1977) 55-56 (Russian); English transl. in Functional Anal. Appl. 11 (1977) 45–47.
[49] F.A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
[50] T. Voronov, Geometric integration theory on supermanifolds, Sov. Sci. Rev. C. Math. Phys. 9 (1992) 1-138.
[51] D. Hernández Ruipérez and J. Muñoz Masqué, Construction intrinsèque du faisceau de Berezin d'une variété graduée, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985) 915–918.
[52] M. Rothstein, Integration on noncompact supermanifolds, Trans. Amer. Math. Soc. 299 (1987) 387–396.
[53] M.M. Vinogradov, Coconnections and integral forms, Ross. Akad. Nauk Dokl. 338 (1994) 295–297 (Russian); English transl. in Russian Acad. Sci. Dokl. Math. 50 (1995).